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LAURA AND PETRARCH: AN INTRIGUING CASE OF CYCLICAL LOVE DYNAMICS*

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Abstract. Three ordinary differential equations are proposed to model the dynamics of love between Petrarch, a celebrated Italian poet of the 14th century, and Laura, a beautiful but married lady. The equations are nonlinear but can be studied through the singular perturbation approach if the inspiration of the poet is assumed to have very slow dynamics. In such a case, explicit conditions are found in the appeals of Laura and Petrarch and in their behavioral parameters that guarantee the existence of a globally stable slow-fast limit cycle. These conditions are consistent with the relatively clear portrait of the two personalities one gets while reading the poems addressed to Laura. On the basis of the scarce and only qualitative information available, the calibration of the parameters is also performed; the result is that the calibrated model shows that the poet's emotions followed for about 20 years a quite regular cyclical pattern ranging from the extremes of ecstasy to despair. All these findings agree with the recent results of Frederic Jones, who, through a detailed stylistic and linguistic analysis of the poems inspired by Laura, has discovered Petrarch's emotional cycle in a fully independent way.

Key words. love dynamics, dynamical system, singular perturbation, cycles

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1. Introduction.

Since differential calculus was introduced by Newton, dynamic phenomena in physics, biology, economics, and all other sciences have been extensively studied with differential equations. Surprisingly, one of the most important problems concerning our lives, the dynamics of love, has not yet been tackled in this way, the only exception being a contribution by Strogatz [1] who reports (in a one-page paper entitled "Love Affairs and Differential Equations") his success in teaching harmonic oscillators by making reference to Romeo and Juliet (see also [2], [3]).

This paper is devoted to the presentation and analysis of a model based on differential equations that pretend to describe the dynamics of love between two persons by accounting for their personalities. The model makes specific reference to a very special and famous case of unrequited love.

The study has been stimulated by a recent work by Frederic Jones [4] on Petrarch's *Canzoniere*, the most celebrated book of love poems in the Western world. A detailed linguistic and stylistic analysis of all the dated poems addressed by Petrarch to his platonic mistress Laura has allowed Jones to conjecture that the poet's emotions follow for approximately 20 years a quite regular cyclical pattern, ranging from the extremes of ecstasy to despair. On the basis of this conjecture, Jones was able to put all undated poems in chronological order, and then show that Petrarch's lyrical style had evolved from early medieval symbolic practices to modern humanistic and artistic modes of thought and expression.

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The analysis presented in this paper is fully independent of that carried out by Jones, but strongly supports his findings. In fact, for suitable values of Laura's and Petrarch's behavioral parameters, the model has a unique attractor which is, indeed, a limit cycle. The conditions required for the existence of such a limit cycle are easily interpretable and agree with the relatively clear portrait of the two personalities emerging from the *Canzoniere*.

These conditions are derived by decomposing the system into fast (Laura and Petrarch's love) and slow (Petrarch's poetic inspiration) components. The analysis shows that small variations of poetic inspiration can induce catastrophic transitions of the fast components. These findings provide extra support to the work of Jones who, on a purely intuitive basis, has advocated catastrophe theory to support his methodology of investigation.

2. Petrarch's emotional cycle. Francis Petrarch (1304–1374), arguably the most lovesick poet of all time, is the author of the *Canzoniere*, a collection of 366 poems (sonnets, songs, sestinas, ballads, and madrigals). In Avignon, at the age of 23, he met Laura, a beautiful but married lady. He immediately fell in love with her and, although his love was not reciprocated, he addressed more than 200 poems to her over the next 21 years. The poems express bouts of ardor and despair, snubs and reconciliations, and they mark the birth of modern love poetry. The verse has influenced countless poets, including Shakespeare.

Unfortunately, only a few lyrics of the *Canzoniere* are dated; the rest are collected in a bafflingly obscure order. The knowledge of the correct chronological order of the poems is a prerequisite for studying the lyrical, psychological, and stylistical development of any poet. This fact is particularly relevant for Petrarch, who somehow represents or, at least, interprets the spectacular transition from the Middle Ages to Humanism. For this reason, the identification of the chronological order of the poems of the *Canzoniere* has been for centuries a problem of major concern for scholars.

Frederic Jones has described in his recent book how he has solved the ordering problem. First, he has noticed that in a number of lyrics Petrarch makes reference to the recurrent nature of his amorous experience. For example, in sonnet LXXVI he says (here and in the following quotations the English version is taken from a forthcoming English translation of the *Canzoniere* by Frederic Jones)

> Amor con sue promesse lunsingando
> mi *ri*condusse alla prigione antica
>
> [Love's promises so softly flattering me
> have led me *back* to my old prison's thrall]

while in sonnet CCXXI he asks

> Quale mio destin, qual forza o qual inganno
> mi *ri*conduce disarmato al campo
> là 've sempre son vinto?
>
> [What fate, what power or what insidiousness
> still guides me *back*, disarmed, to that same field
> wherein I'm always crushed?]

Also Laura's attitude is reported to have recurrently softened. For example, in ballad CXLIX Petrarch says

Di tempo in tempo mi si fa men dura
l'angelica figura e'l dolce riso,
et l'aria del bel viso
e degli occhi leggiadri meno oscura

[*From time to time* less reproachful seem to me
her heavenly figure, and her charming face,
and sweet smile's airy grace,
while her dancing eyes grow far less dark I see]

Second, Jones has collected all dated poems written when Laura was alive. These amount to 42, but only 23 have a fairly secure date. The first (sonnet X) was written in 1330 and the last (sonnet CCXII) in 1347. Then, Jones has thoroughly analyzed each of these 23 poems from a linguistic and lyrical point of view. On the basis of this analysis (described in detail in the third chapter of his book) he has assigned a grade ranging from -1 to $+1$ to each poem. The maximum grade ($+1$) stands for ecstatic love, while very negative grades correspond to deep despair, as in sonnet LXXIX, where Petrarch says

Così mancando vo di giorno in giorno,
sì chiusamente, ch'i' sol me ne accorgo
et quella che guardando il cor mi strugge.

[Therefore my strength is ebbing day by day,
which I alone can secretly survey,
and she whose very glance will scourge my heart.]

Intermediate grades indicate less extreme feelings like ardor, serene love, friendship, melancholy, and anguish. For example, sonnet CLXXVI, where Petrarch says

Parme d'udirla, udendo i rami et l'ore
et le frondi, et gli augei lagnarsi, et l'acque
mormorando fuggir per l'erba verde.

[Her I seem to hear, hearing bough and wind's caress,
as birds and leaves lament, as murmuring flees
the streamlet coursing through the grasses green.]

is graded -0.45 (corresponding to melancholy).

The dates and the grades of the 23 poems are reported as points in Fig. 1 (extracted from [4]), together with four rising segments of a dotted line interpolating some of the points. The distances between pairs of nearby segments are not very different, so that one is naturally brought to imagine that these segments are "fragments" of a cyclical pattern. In other words, the analysis indicates that Petrarch's emotions have varied almost periodically over time and that the period of this emotional cycle is slightly less than four years.

This discovery has been fully exploited by Jones. Indeed, he has extrapolated with a naïve technique the segments of Fig. 1, thus deriving an almost cyclical graph $E(t)$ describing the time pattern of Petrarch's emotions during the entire interval concerned [1330, 1348]. Then, he has given a grade $\hat{E}$ to each undated poem and derived a set $\{t_i\}$ of potential dates for each sonnet by setting $E(t_i) = \hat{E}$. Finally, using historical and other information about Petrarch's life and his visits from Avignon to Italy, he has eliminated all but one potential date for each poem, thus solving the ordering

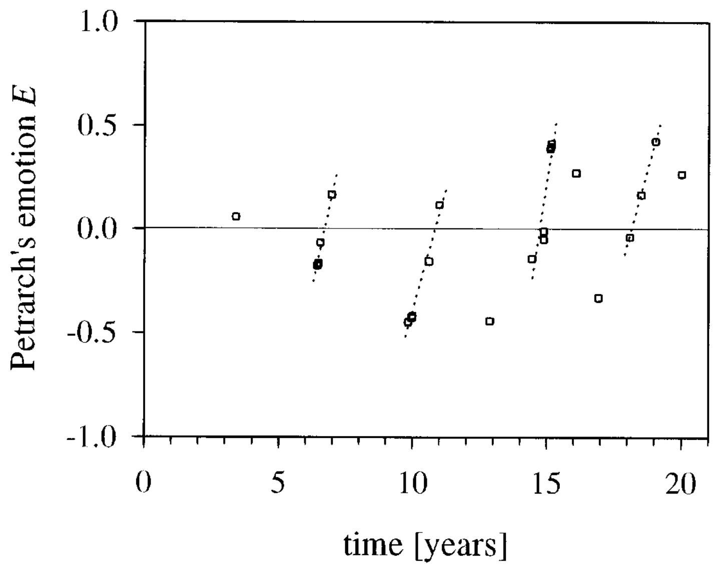

FIG. 1. *The coordinates of the points are the dates and the grades of the* 23 *poems analyzed by Jones* [4]. *The dotted lines are fragments of Petrarch's emotional cycle.* (*Extracted from* [4].)

problem completely.

The key point of the study is undoubtedly the discovery of Petrarch's emotional cycle. But, in a sense this discovery is only a conjecture supported by relatively weak arguments. Indeed, it is based on noisy data, because each point of Fig. 1 is affected by a horizontal error due to the uncertainty of the date of the poem, and by a vertical error due to Jones's subjective evaluation of the poet's emotional states. Independent arguments supporting the same conjecture, like those given in this paper, are therefore important for reinforcing the conjecture.

3. A model of Laura and Petrarch. The emotions of Laura and Petrarch are now modeled by means of three ordinary differential equations. Laura is described by a single variable $L(t)$, representing her love for the poet at time t. Positive and high values of L mean warm friendship, while negative values should be associated with coldness and antagonism. The personality of Petrarch is more complex; its description requires two variables: $P(t)$, love for Laura, and $Z(t)$, poetic inspiration. High values of P indicate ecstatic love, while negative values stand for despair.

The model is the following:

$$\frac{dL(t)}{dt} = -\alpha_1 L(t) + R_L(P(t)) + \beta_1 A_P, \tag{1}$$

$$\frac{dP(t)}{dt} = -\alpha_2 P(t) + R_P(L(t)) + \beta_2 \frac{A_L}{1+\delta Z(t)}, \tag{2}$$

$$\frac{dZ(t)}{dt} = -\alpha_3 Z(t) + \beta_3 P(t), \tag{3}$$

where $R_L(\cdot)$ and $R_P(\cdot)$ are *reaction functions* specified below, A_P [A_L] is the *appeal* (physical, as well as social and intellectual) of Petrarch [Laura], and all greek letters are positive constant parameters (this means that variations in the personalities of Laura and Petrarch due to aging or other external factors are not considered).

The rate of change of the love of Laura (eq. (1)) is the sum of three terms. The first, negative for positive L, describes the *forgetting process* characterizing each individual. The second, namely $R_L(P)$, is the reaction of Laura to the love of Petrarch, while the third is her response to his appeal. Equation (2) is similar to (1) with one relevant exception: the response of Petrarch to the appeal of Laura depends also upon his inspiration Z. This takes into account the well-established fact that high moral tensions, like those associated with artistic inspiration, attenuate the role of the most basic instincts. And there is no doubt that the tensions between Petrarch and Laura are of a passionate nature. For example, in sestina XXII, he says

> Con lei foss'io da che si parte il sole,
> et non ci vedess' altri che le stelle,
> sol una nocte, et mai non fosse l'alba;
>
> [Would I were with her when first sets the sun,
> and no one else could see us but the stars,
> one night alone, and it were never dawn;]

while in his *Posteritati* he confesses (in Latin): "Libidem me prorsus expertem dicere posse optarem quidem, sed si dicat mentiar" [I would truly like to say absolutely that I was without libidinousness, but if I said so I would be lying]. Finally, (3) simply says that the love of Petrarch sustains his inspiration which, otherwise, would exponentially decay with a time constant $1/\alpha_3$. In other words, poetic inspiration is an exponentially weighted integral of the passion of the poet for his mistress.

The two reaction functions $R_L(P)$ and $R_P(L)$ must now be specified. Since most individuals love to be loved and hate to be hated, the most simplistic choice (see [1]) would be to assume that the reaction functions are linear. The linearity of $R_P(L)$ is undoubtedly acceptable (at least for $L < 0$) since in his poems the poet has very intense reactions to the most relevant signs of antagonism from Laura. Thus, we assume

$$R_P(L) = \beta_2 L. \tag{4}$$

It is to be noted that the use of the proportionality coefficient β_2, already employed in (2) to specify the response of Petrarch to the appeal of Laura, is always possible by measuring A_L in suitable units.

On the other hand, a linear reaction function is not appropriate for Laura. In fact, only close to the origin can $R_L(P)$ be assumed to be linear, thus interpreting the natural inclination of a beautiful high-society lady to stimulate harmless flirtations. But Laura never goes too far beyond gestures of pure courtesy: she smiles and glances. However, when Petrarch becomes more demanding and puts pressure on her, even indirectly when his poems are sung in public, she reacts very promptly and rebuffs him, as described explicitly in a number of poems, like sonnet XXI:

> Mille fiate, o dolce mia guerrera,
> per aver co' begli occhi vostri pace
> v'aggio proferto il cor; mâ voi non piace
> mirar sí basso colla mente altera.
>
> [A thousand times, o my sweet enemy,
> to come to terms with your enchanting eyes

I've offered you my heart, yet you despise
aiming so low with mind both proud and free.]

This suggests the use of a reaction function $R_L(P)$ which, for $P > 0$, first increases and then decreases. But the behavior of Laura is also nonlinear for negative values of P. In fact, when $P \ll 0$, i.e., when the poet despairs, Laura feels very sorry for him. Following her genuine Catholic ethic she arrives at the point of overcoming her antagonism by strong feelings of pity, thus reversing her reaction to the passion of the poet. This behavioral characteristic of Laura is repeatedly described in the *Canzoniere*. For example, in sonnet LXIII the poet says

Volgendo gli occhi al mio novo colore
che fa di morte rimembrar la gente,
pietà vi mosse; onde, benignamente
salutando, teneste in vita il core.

[Casting your eyes upon my pallor new,
which thoughts of death recalls to all mankind,
pity in you I've stirred; whence, by your kind
greetings, my heart to life's kept true.]

The above is equivalent to saying that the function $R_L(P)$, besides having a positive maximum for $P > 0$, has a negative minimum for $P < 0$.

In the following, Laura's reaction function $R_L(P)$ is assumed to be a cubic function, i.e.,

$$R_L(P) = \beta_1 P \left(1 - \left(\frac{P}{\gamma}\right)^2\right), \tag{5}$$

where β_1 in this equation is justified as β_2 in (4). Thus, for $P = \gamma$ flattery compensates for antagonism (so that $R_L(\gamma) = 0$), while for $P = -\gamma$ antagonism is compensated for by pity. Moreover, the value of P for which the reaction is maximum [minimum] is $\gamma/\sqrt{3}$ [$-\gamma/\sqrt{3}$]. Obviously, in no way can one support, from the *Canzoniere* and its related historical information, the specific choice (5) for Laura's reaction function. It is better to confess that this choice is due to convenience, since it allows one to derive analytically a number of interesting results. Nevertheless, these results are robust, in the sense that they also hold for other reaction functions, obtained from (5) through reasonable perturbations. This point will be discussed later.

In conclusion, if one takes into account (4), (5), the Laura and Petrarch model (1)–(3) becomes

$$\frac{dL}{dt} = -\alpha_1 L + \beta_1 \left[P\left(1 - \left(\frac{P}{\gamma}\right)^2\right) + A_P\right], \tag{6}$$

$$\frac{dP}{dt} = -\alpha_2 P + \beta_2 \left[L + \frac{A_L}{1 + \delta Z(t)}\right], \tag{7}$$

$$\frac{dZ}{dt} = -\alpha_3 Z + \beta_3 P. \tag{8}$$

This is the model to be discussed in what follows. It is important to notice that this model encapsulates the personalities of Laura and Petrarch as they emerge from all the poems of the *Canzoniere*, and not from the particular 23 poems analyzed by

Jones. Thus, the model and the conclusions one can draw from it are “independent” from those (see Fig. 1) obtained empirically by Jones.

4. The *L-P* cycle. In principle, model (6)–(8) describes the dynamics of the emotions evolving between any poet and his reluctant lady. Obviously, each pair is characterized by specific appeals and behavioral parameters. In the case of Laura and Petrarch the calibration of the parameters is particularly difficult and highly subjective: all available information comes from a series of poems. Therefore the values given to the parameters in this section do not pretend to be the “correct” ones. They are only my personal estimates based on the feelings and impressions I had when reading the *Canzoniere*.

Let us start with the parameters α_i, $i = 1, 2, 3$, that describe the forgetting processes. As for α_1 and α_2, there is no doubt that

$$\alpha_1 > \alpha_2.$$

Indeed, Laura never appears to be strongly involved, while the poet definitely has a tenacious attachment. This is described in a number of lyrics, as in sonnet XXXV:

Solo et pensoso i più deserti campi
vo mesurando a passi tardi e lenti,
. . .
Ma pur sì aspre vie ne’ sí selvagge
cercar non so ch’Amor non venga sempre
ragionando con meco, et io co llui.

[Alone and lost in thought, each lonely strand
I measure out with slow and laggard step,
. . .
Yet I cannot find such harsh and savage trails
where love does not pursue me as I go,
with me communing, as with him do I.]

On the other hand,

$$\alpha_2 > \alpha_3,$$

since the inspiration of the poet wanes very slowly. Indeed, Petrarch continues to write (over one hundred poems) for more than ten years after the death of Laura. The main theme of these lyrics is not his passion for Laura, which has long since faded, but the memory for her and the invocation of death. This is clear, for example, in song CCLXVIII, written about two years after her demise, when he says

. . .
Tempo è ben di morire,
et ò tardato più ch’i non vorrei.
Madonna è morta, et à seco il mio core;
e volendol seguire,
interromper conven quest’anni rei,
perché mai veder lei
di qua non spero, et l’aspettar m’è noia.

[. . .
It’s time indeed to die,

and I have lingered more than I desire.
My lady's dead, and with her my heart lies;
and, keen with her to fly,
I now would from this wicked world retire,
since I can no more aspire
on earth to see her, and delay will me destroy.]

Consistently, we will constrain the forgetting coefficients α_i to satisfy the relationships

$$\alpha_1 = 3\alpha_2 \qquad \alpha_3 = \frac{1}{10}\alpha_2. \tag{9}$$

For example, the triplet

$$\alpha_1 = 3, \qquad \alpha_2 = 1, \qquad \alpha_3 = 0.1$$

satisfies (9) and can be interpreted by imagining that Laura forgets Petrarch in about four months and that Petrarch's passion fades in one year, whereas he remains inspired for ten years.

As far as the reaction parameters β_i are concerned, again Laura is assumed to be much less sensitive than Petrarch, i.e.,

$$\beta_1 = \alpha_2, \qquad \beta_2 = 5\alpha_2, \qquad \beta_3 = 10\alpha_2. \tag{10}$$

This is equivalent to saying that her time of reaction $1/\beta_1$ equals the forgetting time of Petrarch, who is five times more reactive than her to love and appeal.

Moreover, we assume that

$$\gamma = \delta = 1, \tag{11}$$

since this is always possible by suitably scaling P and Z.

Finally, opposite signs are given to the appeals of Laura and Petrarch (see Fig. 2), namely,

$$A_L = 2, \qquad A_P = -1. \tag{12}$$

Indeed, as repeatedly described in the *Canzoniere*, she is a beautiful and inspiring lady. By contrast, Petrarch is a cold scholar interested in history and letters. He is appointed "*cappellanus continuus commensalis*" by Cardinal Giovanni Colonna, and this ecclesiastic appointment brings him frequently to Avignon, where Laura lives. The negativity of the appeal of Petrarch is somehow recognized by the poet himself, who, in sonnet XLV, while talking about Laura's mirror, says

Il *mio adversario* in cui veder solete
gli occhi vostri ch'Amore e 'l ciel honora,

[*My rival* in whose depths you're wont to see
your own dear eyes which Love and heaven apprize,]

Once Petrarch's forgetting coefficient α_2 is fixed, (9)–(12) produce one complete parameter setting. A broad estimate for α_2 is $\alpha_2 = 1$, corresponding to a time constant for Petrarch's love of one year and to the following parameter setting:

$$\begin{array}{lll} \alpha_2 = 3, & \alpha_2 = 1, & \alpha_3 = 0.1, \\ \beta_1 = 1, & \beta_2 = 5, & \beta_3 = 10, \\ \gamma = \delta = 1, & A_L = 2, & A_P = -1. \end{array} \tag{13}$$

FIG. 2. *Portraits of Laura and Petrarch (Biblioteca Medicea Laurenziana, ms. Plut. cc.* VIIIv–IX*, Florence, Italy; courtesy of Ministero per i Bene Culturali e Ambientali).*

Equations (6)–(8) with the parameter values (13) can be numerically integrated for a period of 21 years, starting on the day (April 6, 1327) when Laura and Petrarch met for the first time and ending on the day she died (April 6, 1348). The selected initial conditions are

$$L(0) = 0, \qquad P(0) = 0, \qquad Z(0) = 0.$$

The first two are obvious, but also the third is plausible since Petrarch has not written any relevant lyric before 1327. The results of the numerical integration (shown in Fig. 3) are qualitatively in full agreement with the *Canzoniere* and with the analysis of Frederic Jones. After a first high peak, Petrarch's love tends toward a regular cycle characterized by alternate positive and negative peaks. Also, $L(t)$ and $Z(t)$ tend towards a cyclic pattern. At the beginning, Petrarch's inspiration rises much more slowly than his love and then remains positive during the entire period. This might explain why Petrarch writes his first poem more than three years after he has met her, but then continues to produce lyrics without any significant interruption. By contrast, Laura's love is always negative. This is in perfect agreement with the *Canzoniere*, where Laura is repeatedly described as adverse. For example, in sonnet XXI, the poet calls her "dolce mia guerrera" [my sweet enemy], while in sonnet XLIV he says

né lagrima però discese anchora
da' be' vostr'occhi, ma disdegno et ira.

[and still no tears your lovely eyes assail,
nothing as yet, but anger and disdain.]

A comparison of Fig. 1 with Fig. 3 shows that the period of the simulated cycle is slightly longer (about 20%) than that identified by Jones. I have therefore increased

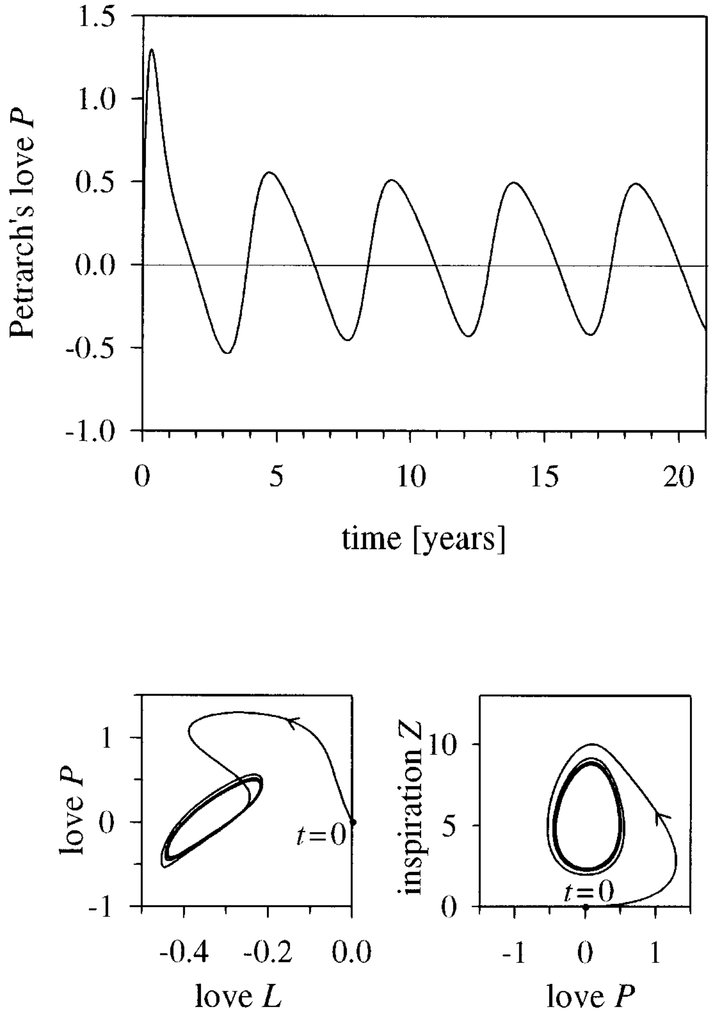

FIG. 3. *Time evolution of Petrarch's love and projections of the trajectory of system* (6)–(8) *with zero initial conditions and parameter values* (13).

α_2 by 20% and repeated the simulation for the new parameter setting:

$$\begin{array}{lll} \alpha_1 = 3.6, & \alpha_2 = 1.2, & \alpha_3 = 0.12, \\ \beta_1 = 1.2, & \beta_2 = 6, & \beta_3 = 12, \\ \gamma = \delta = 1, & A_L = 2, & A_P = -1. \end{array} \quad (14)$$

Because of (9)–(12) the new solution can be simply obtained by stretching the old one in time by 20%. The new time pattern for Petrarch's love is shown in Fig. 4 together with the grades given by Jones to the 23 dated poems (see Fig. 1); obviously, I have assumed that the grade is proportional to Petrarch's love. The fit is very good—actually, as good as that which is usually obtained when calibrating models of electrical and mechanical systems. Moreover, the fit could be further improved by slightly modifying the parameter values. But I do not show results along this line, because I do not want to give the impression that I believe that Petrarch had been

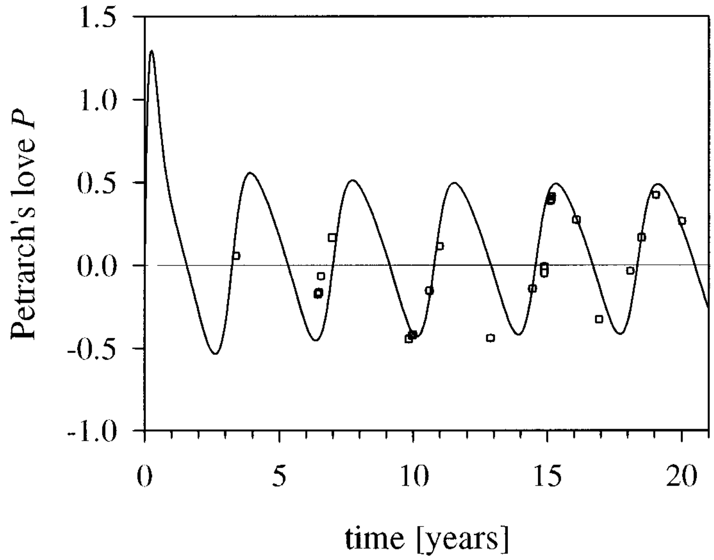

FIG. 4. *Time evolution of Petrarch's love computed with model* (6)–(8) *and parameter values* (14). *The points are the estimates of Jones* [4] *concerning* 23 *dated poems* (*see Fig.* 1).

producing his lyrics like a rigid, deterministic machine. Nevertheless, we can conclude that the *L*-*P* model with the parameter setting (14) strongly supports Frederic Jones's conjecture.

In order to complete the analysis, I have tested the robustness of the *L*-*P* cycle with respect to perturbations of the parameters. For this, the package LOCBIF, a professional software package for the analysis of the bifurcations of continuous-time dynamical systems, has been used. The software is based on a continuation technique described in Khibnik et al. [5]. By varying only one parameter at a time (except γ and δ) with respect to the reference values indicated in (14), I have found that the cycle eventually disappears through a supercritical Hopf bifurcation. This means that when the parameters are varied, the *L*-*P* cycle can shrink and finally be replaced by a stable equilibrium point. The percentage variations of the parameters giving rise to this Hopf bifurcation are the following:

	α_1	α_2	α_3	β_1	β_2	β_3	A_L	A_P
reference value	3.6	1.2	0.12	1.2	6	12	2	-1
% variation	36	23	134	-22	-19	-82	-53	45

The most critical parameters are α_2, β_1, and β_2, but we need to vary them quite consistently to transform the cyclical regime into a stationary one. This proves, numerically, that the *L*-*P* cycle we have identified is not due to a perverse combination of the parameters, but is, on the contrary, quite robust.

Finally, I have also tested the robustness of the cycle with respect to functional perturbations. For this, I have simulated the system obtained by slightly modifying the reaction functions of Laura and Petrarch (see (4) and (5)). In particular, I have checked that the *L*-*P* cycle does not disappear if the reaction functions are not symmetric (this is obvious for perturbations of Petrarch's reaction function (4), since Laura's love is always negative on the *L*-*P* cycle (see Fig. 3)).

5. Slow-fast limit cycles. We now abandon the specific case of Laura and Petrarch (see parameter setting (14)) with the aim of deriving general conditions on the parameters that guarantee the existence of a limit cycle. For this, let us rewrite the model as

$$\frac{dL}{dt} = -\alpha_1 L + \beta_1 \left[P \left(1 - \left(\frac{P}{\gamma} \right)^2 \right) + A_P \right], \tag{6}$$

$$\frac{dP}{dt} = -\alpha_2 P + \beta_2 \left[L + \frac{A_L}{1 + \delta Z(t)} \right], \tag{7}$$

$$\frac{dZ}{dt} = \varepsilon[-Z + \mu P] \tag{15}$$

by slightly changing the form of (8). Note that the two new parameters ε and μ are positive ($\varepsilon = \alpha_3$, $\mu = \beta_3/\alpha_3$).

If ε is small, the variable $Z(t)$ is slow with respect to $L(t)$ and $P(t)$, so that the *singular perturbation method* [6], [7] can be used. Roughly speaking, the method affirms that the system can be decomposed into fast and slow components (in the present case the fast subsystem is described by (6), (7) with $Z = constant$, and the slow one by (15)). Such a decomposition allows one to construct the so-called *singular orbits*, which are composed of concatenations of alternate fast and slow transitions. The main result of singular perturbation theory is that, under suitable regularity assumptions (which are satisfied in the present case) any orbit of the system approaches, for $\varepsilon \to 0$, the corresponding singular orbit. This result has been proved first by Tikhonov [8] for the simple case of singular orbits not passing through bifurcation points of the fast subsystem and by Pontryagin [9] for the case of singular orbits passing through nondegenerate (i.e., quadratic) fold bifurcations of the fast subsystem. Remarks on the use of this method for detecting limit cycles in second- and third-order systems can be found in Muratori and Rinaldi [10], [11], while examples of applications can be found in [12], [13], [14].

The fast subsystem. Equations (6), (7), with Z frozen to a constant value, describe the fast dynamics of the system. The state $(L(t), P(t))$ of such a system cannot tend towards a limit cycle since the divergence is negative (Dulac's criterion [3]). Thus, $L(t)$ and $P(t)$ tend towards an equilibrium, namely, towards a constant solution (L, P) of (6), (7).

By eliminating L from (6), (7) with $dL/dt = dP/dt = 0$, and separating the variables P and Z, one obtains the equation

$$\Phi(Z) = \Psi(P), \tag{16}$$

where

$$\Phi(Z) = -\gamma^2 \left(A_P + \frac{\alpha_1}{\beta_1} \frac{A_L}{1 + \delta Z} \right), \tag{17}$$

$$\Psi(P) = 3\Delta P - P^3$$

with

$$\Delta = \frac{\beta_1 \beta_2 - \alpha_1 \alpha_2}{3 \beta_1 \beta_2} \gamma^2. \tag{18}$$

If Δ is positive, i.e., if

$$\beta_1 \beta_2 > \alpha_1 \alpha_2, \tag{19}$$

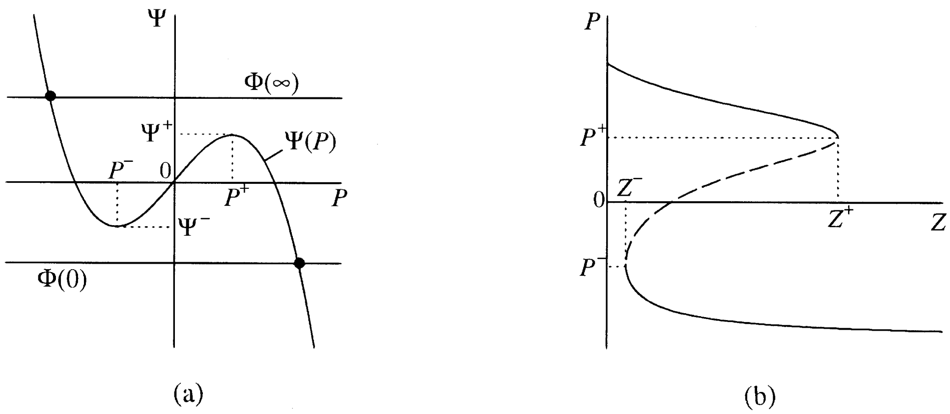

FIG. 5. (a) *The graph of the function* $\Psi(P)$ *when conditions* (19)–(21) *hold (the two dots are the solutions of* (16) *for* $Z = 0$ *and* $Z = \infty$*);* (b) *The* P*-component of the equilibrium of the fast subsystem for constant values of* Z*: in the interval* (Z^-, Z^+)*, there are three equilibria, two stable (continuous) lines, and one unstable (dashed) line.*

the function $\Psi(P)$ has a minimum $\Psi^- = -2\Delta\sqrt{\Delta}$ [maximum $\Psi^+ = 2\Delta\sqrt{\Delta}$] at point $P^- = -\sqrt{\Delta}$ [$P^+ = \sqrt{\Delta}$].

The function $\Psi(P)$ is plotted in Fig. 5a together with the two horizontal straight lines $\Phi(Z)$ corresponding to the two extreme values of poetic inspiration, namely, $Z = 0$ and $Z = \infty$. The graph in Fig. 5a refers to the case in which $\Phi(\infty) > \Psi^+$ and $\Phi(0) < \Psi^-$, namely, to the case in which the following two inequalities among the parameters hold:

$$A_P < -\frac{2\Delta\sqrt{\Delta}}{\gamma^2}, \tag{20}$$

$$A_L > \frac{\beta_1}{\alpha_1}\left(2\frac{\Delta\sqrt{\Delta}}{\gamma^2} - A_P\right), \tag{21}$$

where Δ is given by (18). Under these conditions, (16) has a unique (and positive [negative]) solution P for $0 \le Z < Z^-$ [$Z > Z^+$], where Z^- [Z^+] is the value of Z for which $\Phi(Z^-) = \Psi^-$ [$\Phi(Z^+) = \Psi^+$]. On the other hand, in the interval (Z^-, Z^+), (16) has three solutions, as shown in the graph of Fig. 5b. The corresponding equilibrium values for L are given by (see (7) with $dP/dt = 0$)

$$L = \frac{\alpha_2}{\beta_2}P - \frac{A_L}{1 + \delta Z}.$$

Thus, when conditions (19)–(21) are satisfied, the fast subsystem (6), (7) with $Z =$ *constant* has a unique equilibrium for small ($Z < Z^-$) and for high ($Z > Z^+$) values of poetic inspiration and three equilibria for intermediate values ($Z^- < Z < Z^+$). The stability of these equilibria can be easily studied by means of the Jacobian matrix

$$J = \begin{vmatrix} -\alpha_1 & \beta_1\left(1 - 3\left(\frac{P}{\gamma}\right)^2\right) \\ \beta_2 & -\alpha_2 \end{vmatrix}. \tag{22}$$

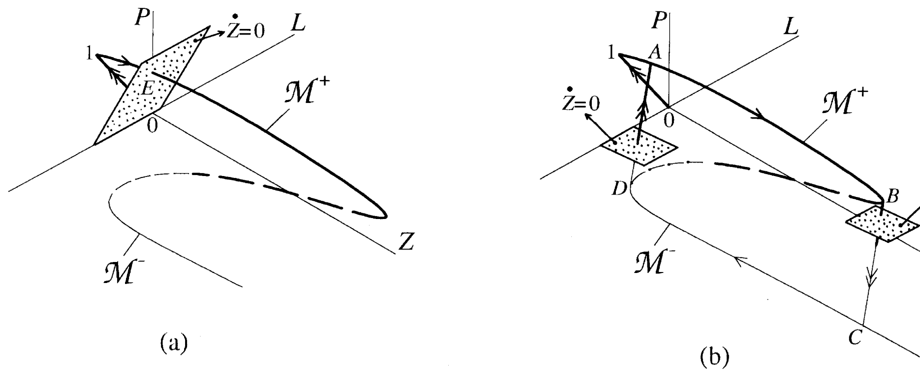

FIG. 6. *The equilibrium manifold of the fast subsystem and its stable branches $\mathcal{M}^+$ (above plane (L,Z)) and $\mathcal{M}^-$ (below plane (L,Z)).* (a) $\mu < \mu_{crit}$*: all singular orbits tend towards the equilibrium point E;* (b) $\mu > \mu_{crit}$*: all singular orbits tend towards the singular limit cycle ABCD.*

The result is that the equilibrium with $P^- < P < P^+$ is unstable (dashed lines in Figs. 5b and 6) while the two others are stable. In the following, the manifold (line) of points $(\overline{L}, \overline{P}, \overline{Z})$, such that $(\overline{L}, \overline{P})$ is an equilibrium of the fast subsystem for $Z = \overline{Z}$, will be called the *equilibrium manifold of the fast subsystem* and denoted by $\mathcal{M}$. Moreover, the upper and lower stable branches of this manifold will be denoted by $\mathcal{M}^+$ and $\mathcal{M}^-$, respectively (see Fig. 6).

Singular orbits and slow-fast limit cycles. We can now consider the slow component of the system, namely, the poetic inspiration Z described by (15) with ε small and positive. In the three-dimensional state space (L, P, Z) the manifold $dZ/dt = 0$ is the plane

$$Z = \mu P.$$

Thus, for μ greater than a critical value μ_{crit}, such a plane separates the two stable branches $\mathcal{M}^+$ and $\mathcal{M}^-$ of the equilibrium manifold $\mathcal{M}$ of the fast subsystem, as shown in Fig. 6b. The critical value of μ is Z^+/P^+ (see Fig. 5b), where Z^+ is the solution of (16) with $P = P^+$. Taking into account (17), and recalling that $P^+ = \sqrt{\Delta}$ and $\Psi(P^+) = 2\Delta\sqrt{\Delta}$, one can easily derive that

$$\mu_{crit} = \frac{1}{\delta\sqrt{\Delta}} \frac{\frac{\alpha_1}{\beta_1} A_L + A_P + 2\frac{\Delta\sqrt{\Delta}}{\gamma^2}}{-\left(A_P + 2\frac{\Delta\sqrt{\Delta}}{\gamma^2}\right)}.$$

Fig. 6 shows the singular orbit starting from the origin in two cases: (a) $\mu < \mu_{crit}$, and (b) $\mu > \mu_{crit}$. In both cases the first segment of the singular orbit is a fast transition from the origin to point $1 \in \mathcal{M}^+$. Since at that point $dZ/dt > 0$, the singular orbit continues with a slow motion along $\mathcal{M}^+$ in the direction of increasing Z. If $\mu < \mu_{crit}$ (Fig. 6a) this slow motion terminates at the equilibrium point E, where $dL/dt = dP/dt = dZ/dt = 0$, so that the singular orbit is 0 1 E (note that all other nearby singular orbits also terminate at point E). If, on the contrary, $\mu > \mu_{crit}$ (Fig. 6b), the slow motion develops along the whole upper branch $\mathcal{M}^+$ of the equilibrium manifold and terminates at point B where the fast subsystem has a fold bifurcation point. Since this fold bifurcation is not degenerate (easy to check),

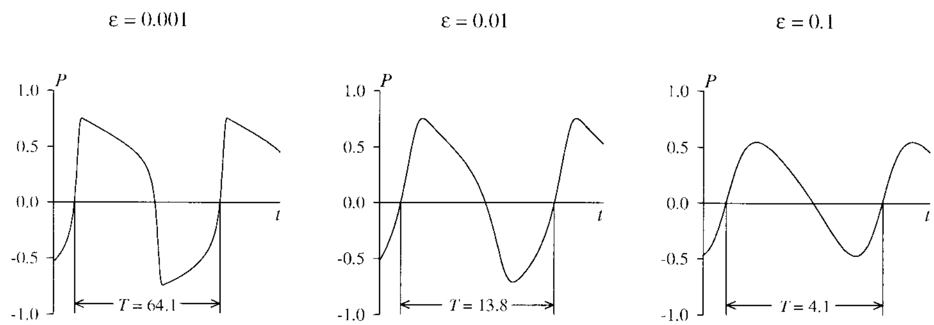

FIG. 7. *Periodic evolution of Petrarch's love for different values of* $\varepsilon\,(=\alpha_3)$. *Parameters are at their reference values* (14) *and* $\mu = 100$. *Notice the high sensitivity of the period* T *of the cycle to* ε.

point B is the starting point of a third segment of the singular orbit, which is a catastrophic transition of the fast subsystem ending at point $C \in \mathcal{M}^-$. Since at that point $dZ/dt < 0$, the singular orbit continues (with slow motion) along $\mathcal{M}^-$ in the direction of decreasing Z. At the fold point D, the slow motion terminates and a fast jump brings the system back to point A on $\mathcal{M}^+$, thus closing a *singular cycle A B C D*. Note that this singular cycle is unique and attracts all nearby singular orbits.

By taking into account the main result of singular perturbation theory, the preceding discussion can be summarized as follows.

THEOREM. *If conditions* (19)–(21) *are satisfied and* ε *is sufficiently small, system* (6), (7), (15) *has an attractor which is an equilibrium if* $\mu < \mu_{crit}$ *and a limit cycle if* $\mu > \mu_{crit}$. *Moreover, the limit cycle tends for* $\varepsilon \to 0$ *to the singular limit cycle A B C D depicted in Fig.* 6b.

Of course, if $\mu < \mu_{crit}$ but ε is not very small, there is no guarantee that the limit cycle resembles the singular limit cycle. Figure 7 shows, for example, the sensitivity with respect to ε of the limit cycle corresponding to the reference parameter setting (14) (which satisfies conditions (19)–(21) and has $\mu > \mu_{crit}$). For $\varepsilon = 0.001$ the slow-fast character of the limit cycle is clearly recognizable, while for $\varepsilon = 0.1$ the fast transitions are not detectable any more. Increasing ε further, the limit cycle might even disappear, through a supercritical Hopf bifurcation. Actually, this is the only bifurcation of the limit cycle I have been able to detect numerically, by means of the package LOCBIF. Figure 8 is a typical example of the application of this software: a two-dimensional parameter space (the space (ε, μ) in the present case) is subdivided into two regions where the asymptotic behavior of the system is either cyclic or stationary. In view of the above theorem, the Hopf bifurcation curve starts from the point μ_{crit} on the vertical axis and is orthogonal to it at that point. Point R in Fig. 8 corresponds to the reference parameter setting (14). As already indicated, if one starts from this point and decreases β_3 (i.e., decreases $\mu = \beta_3/\alpha_3$) or increases α_3 (i.e., increases $\varepsilon = \alpha_3$ and decreases $\mu = \beta_3/\alpha_3$), the limit cycle shrinks and disappears.

Interpretation of the results. It is worth noticing that conditions (19)–(21) and $\mu > \mu_{crit}$ that guarantee the existence of a slow-fast limit cycle (see the above theorem) can be simply interpreted in terms of the main characteristics of the two lovers. Condition (19) says that the two individuals are very sensitive since the product of

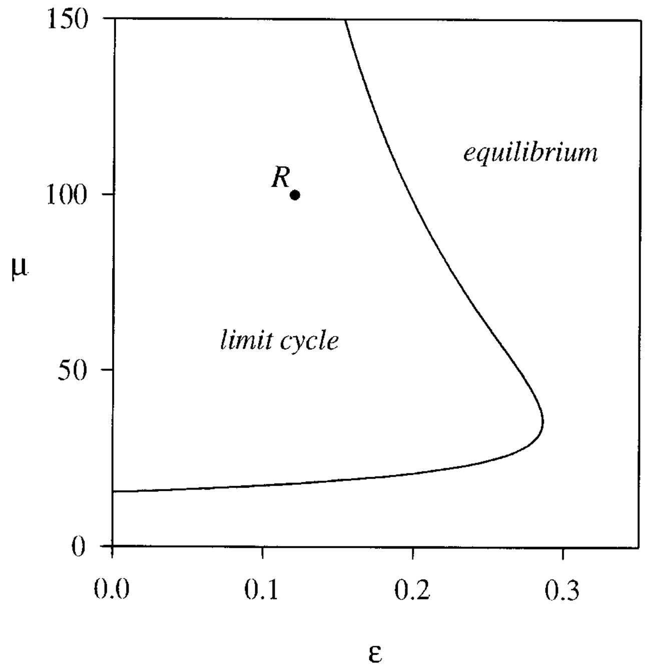

FIG. 8. *Supercritical Hopf bifurcation curve in the parameter space* (ε, μ). *Point R corresponds to the reference parameter values* (14).

their reactiveness exceeds the product of their forgetting coefficients. More rigorously, condition (19) implies that the Jacobian (22) of the fast subsystem is unstable at the origin. This means that in the early phase of their relationship the two individuals form an "explosive" couple. Condition (20) says that the poet is definitely not appealing to the lady, while condition (21) says just the opposite for the lady to him. Finally, condition $\mu > \mu_{crit}$ states that the poet is very effectively inspired by his love for the lady. All these characteristics are present in the case of Laura and Petrarch and, indeed, our reference parameter setting (14) satisfies conditions (19)–(21) and $\mu > \mu_{crit}$.

6. Concluding remarks. In this paper a minimal model of love dynamics between a poet and a lady has been proposed and adapted to the very special and famous case of Laura and Petrarch. The merits of this study are three. The model is one of the rare three-dimensional models to which the singular perturbation method has been successfully applied in a fully analytical way (see [12], [13], [14] for other examples). The study formally explains why Laura and Petrarch have been caught in a never-ending cyclical love story. This is in agreement with the purely empirical discovery of Frederic Jones [4]. Finally, the third and possibly most important merit is that I have used for the first time ordinary differential equations for describing in some detail the dynamics of love between two persons. Of course, this possibility could have been argued by imagining that the approach followed by Strogatz [1] in building up his "ideal love oscillator" could be adapted to realistic situations. Filling the gap between arguing and proving that this is possible has been a stimulating challenge.

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REFERENCES

[1] S. H. Strogatz, *Love affairs and differential equations*, Math. Magazine, 61 (1988), p. 35.

[2] M. J. Radzicki, *Dyadic processes, tempestuous relationships, and system dynamics*, System Dynamics Review, 9 (1993), pp. 79–94.

[3] S. H. Strogatz, *Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry and Engineering*, Addison-Wesley, Reading, MA, 1994.

[4] F. J. Jones, *The Structure of Petrarch's Canzoniere*, Brewer, Cambridge, UK, 1995.

[5] A. I. Khibnik, Y. A. Kuznetsov, V. V. Levitin, and E. V. Nikolaev, *Continuation techniques and interactive software for bifurcation analysis of ODEs and iterated maps*, Phys. D, 62 (1993), pp. 360–371.

[6] F. Hoppensteadt, *Asymptotic stability in singular perturbation problems*, J. Differential Equations, 15 (1974), pp. 510–521.

[7] R. E. O'Malley, Jr., *Introduction to Singular Perturbations*, Academic Press, New York, 1974.

[8] A. N. Tikhonov, *Systems of differential equations containing small parameters at derivatives*, Mat. Sb., 31 (1952), pp. 575–586 (in Russian).

[9] L. C. Pontryagin, *Asymptotic behavior of solutions of systems of differential equations with a small parameter at higher derivatives*, Izv. Akad. Nauk. SSSR Ser. Math., 21 (1957), pp. 605–626 (in Russian).

[10] S. Muratori and S. Rinaldi, *Remarks on competitive coexistence*, SIAM J. Appl. Math., 49 (1989), pp. 1462–1472.

[11] S. Muratori and S. Rinaldi, *A separation condition for the existence of limit cycles in slow-fast systems*, Appl. Math. Modelling, 15 (1991), pp. 312–318.

[12] D. Ludwig, D. D. Jones, and C. S. Holling, *Qualitaive analysis of insect outbreak systems: The spruce budworm and forest*, J. Anim. Ecol., 47 (1978), pp. 315–332.

[13] S. Muratori and S. Rinaldi, *Low- and high-frequency oscillations in three-dimensional food chain systems*, SIAM J. Appl. Math., 52 (1992), pp. 1688–1706.

[14] S. Rinaldi and S. Muratori, *Limit cycles in slow-fast forest-pest models*, Theor. Popul. Biol., 41 (1992), pp. 26–43.